

THE NON-VISCOUS BURGERS EQUATION ASSOCIATED WITH RANDOM POSITIONS IN COORDINATE SPACE: A THRESHOLD FOR BLOW UP BEHAVIOUR

SERGIO ALBEVERIO ¹, OLGA ROZANOVA ²

ABSTRACT. It is well known that the solutions to the non-viscous Burgers equation develop a gradient catastrophe at a critical time provided the initial data have a negative derivative in certain points. We consider this equation assuming that the particle paths in the medium are governed by a random process with a variance which depends in a polynomial way on the velocity. Given an initial distribution of the particles which is uniform in space and with the initial velocity linearly depending on the position we show both analytically and numerically that there exists a threshold effect: if the power in the above variance is less than 1, then the noise does not influence the solution behavior, in the following sense: the mean of the velocity when we keep the value of position fixed goes to infinity outside the origin. If however the power is larger or equal 1, then this mean decays to zero as the time tends to a critical value.

1. INTRODUCTION

The non-viscous Burgers equation is perhaps the simplest equation that models the nonlinear phenomena in a force free mass transfer. It has the form

$$u_t + (u, \nabla) u = -\beta u, \quad (1.1)$$

where $u(x, t) = (u_1, \dots, u_n)$ is a vector-function $\mathbb{R}^{n+1} \rightarrow \mathbb{R}^n$, $\beta \geq 0$ is a constant friction coefficient. Consider the Cauchy data

$$u(x, 0) = u_0(x). \quad (1.2)$$

Problem (1.1), (1.2) has an implicit solution

$$u(t, x) = e^{-\beta t} u_0\left(x - \frac{1}{\beta}(e^{\beta t} - 1)u(t, x)\right),$$

for $\beta > 0$ and

$$u(t, x) = u_0(x - tu(t, x)),$$

for $\beta = 0$.

In several cases we can obtain an explicit solution. For example, if

$$u_0(x) = \alpha x, \quad \alpha \in \mathbb{R}, \quad (1.3)$$

one easily gets

$$u(t, x) = \frac{\alpha x e^{-\beta t}}{1 + \frac{\alpha}{\beta}(1 - e^{-\beta t})}, \quad \beta > 0, \quad (1.4)$$

Date: August 5, 2008.

1991 Mathematics Subject Classification. 35R60.

Key words and phrases. Burgers equation, random position of particle, gradient catastrophe.

Supported by DFG 436 RUS 113/823/0-1.

and, for $\beta = 0$:

$$u(t, x) = \frac{\alpha x}{1 + \alpha t}. \quad (1.5)$$

Thus, if $\alpha < -\beta$, the solution develops a singularity at the origin as $t \rightarrow T$, $0 < T < \infty$, where

$$T = \frac{1}{\beta} \ln \frac{\alpha}{\alpha + \beta}, \quad \text{for } \beta > 0, \quad T = -\frac{1}{\alpha}, \quad \text{for } \beta = 0, \quad \alpha \neq 0. \quad (1.6)$$

This phenomenon is called the gradient catastrophe. It is well known (see [1]) that a viscous perturbation of form $\sigma \Delta u$, $\sigma > 0$, entails a globally in time smooth solution (at least for bounded initial data). An exceptional case is exactly given by a solution which is linear in x as mentioned above, which does not feel the viscous term.

Our main question is: can a stochastic perturbation suppress the appearance of unbounded gradients?

We can introduce the Lagrangian coordinate $x(t)$ to label a point which moves together with the medium, that is $\frac{dx(t)}{dt} = u(t, x(t)) := u_1(t)$. Thus, $x = x(t)$ is the equation for the particle path, when the particle moves along the Burgers fluid. Equation (1.1) is equivalent to the following system of ODE:

$$\dot{x}(t) = u_1(t), \quad \dot{u}_1(t) = -\beta u_1(t) \quad (1.7)$$

Further on we will omit the index 1.

In the theory of stochastic dynamical systems one often considers a stochastic perturbation of the velocity, which leads to the appearance of a white noise in the second of equations (1.7). The problem of solving such equations was investigated in many works (see, e.g. [2], [3], [4], [5], [6]). This type of stochastic perturbation corresponds to the stochastically forced Burgers equation, or in the language of physicists, Burgers turbulence. This has been an area of intensive research activity in the last decade (see e.g. [7], and for a very recent review [8], and references therein).

The behavior of the gradient of velocity was studied earlier in other contexts in [9], [10], but this problem is quite different from the problem considered in this paper.

In the present paper we consider a medium with random particles paths, more precisely, described by a $2 \times n$ dimensional Itô stochastic differential system of equations

$$\begin{aligned} dX_k(t) &= U_k(t) dt + \sigma |U(t)|^p d(W_k)_t, \\ dU_k(t) &= -\beta U_k(t) dt, \quad k = 1, \dots, n, \\ X(0) &= x, \quad U(0) = u, \quad t \geq 0, \end{aligned} \quad (1.8)$$

where $(X(t), U(t))$ runs in the phase space $\mathbb{R}^n \times \mathbb{R}^n$, $\sigma > 0$ and $p \geq 0$ are constants, $(W)_t = (W)_{k,t}$, $k = 1, \dots, n$, is the n - dimensional Brownian motion. We remark that for $p \leq 1$ for initial distributions of x and u from the class $L^2(\mathbb{R}^n)$ one can guarantee a global existence of a unique solution to (1.8)[11].

Let us denote by $\hat{u}(t, x)$ the mean of the velocity $U(t)$ at time t when we keep the value of $X(t)$ at time t fixed but allow $U(t)$ to take any value it wants (e.g.[12]) and chose this function for comparison with the solution to the solution of the non-viscous Burgers equation.

We can interpret system (1.8) also as follows: assume that we measure the position of a particle with an error depending on its velocity and then try to restore the velocity. If the coefficient p increases, the error for large velocities increases, too.

It is natural to expect that the mean of difference between two "very indefinite" neighbor coordinates necessary to calculate the velocity tends to zero.

Can we hope to extract from our measurement a realistic information on such critical phenomena as the blow up occurring in a medium described by the Burgers equation associated with (1.8)? As we will see, at least for uniform initial distribution of particles (in the sense of Sec.3, see below) the answer depends on the exponent p , namely, if $p \geq 1$, the information gets lost. The threshold value $p = 1$ is not astonishing. The sub-linear rise of the drift and diffusion ($p \leq 1$) coefficients warrants the global existence of the SPDE solution provided the initial distributions of the particle positions and the velocities are square integrable. Therefore one can hope that the solutions to the SPDE with a sub-linear diffusion coefficient behave in a some "predictable" way.

For example, for $\beta = 0$ the function $\hat{u}(t, x)$ demonstrates the following behaviour near the critical time T . For $p \in [0, 1)$

$$\hat{u}(t, x) = \frac{\alpha}{1 - \frac{t}{T}} x + o\left(\frac{1}{1 - \frac{t}{T}}\right), \quad t \rightarrow T, \quad x \in \mathbb{R}^n;$$

for $p > 1$

$$\hat{u}(t, x) = -C|x|^{\frac{2(1-p)}{p}} x \left(1 - \frac{t}{T}\right) + o\left(1 - \frac{t}{T}\right), \quad t \rightarrow T, \quad x \neq 0,$$

where C is the positive constant depending in particular on the dimension of space, however,

$$\hat{u}(t, x) = \frac{\alpha}{1 - \frac{t}{T}} x + o(|x|), \quad t \in [0, T), \quad x \rightarrow 0.$$

In other words, for $p \in [0, 1)$ we see that $\hat{u}(t, x) \sim u(t, x)$, $t \rightarrow T$ at every point $x \in \mathbb{R}^n$, where $u(t, x)$ is given by (1.5), i.e. $\hat{u}(t, x)$ keeps the property of solutions to the inviscid Burgers to have a gradient catastrophe. For $p \geq 1$ outside the origin $x = 0$ the function $\hat{u}(t, x) \rightarrow 0$, $t \rightarrow T$, and the features of the solution of the non-perturbed equation fail. A jump is being formed at the origin $x = 0$ but the height of this jump at the same time tends to zero as $t \rightarrow T$.

The paper is organized as follows. In Sec.2 we find an exact solution to the Fokker-Planck equation with special initial data and derive an integral formula for \hat{u} . In Sec.3 we consider the simplest case $p = 0$, where for several initial distribution it is possible to find \hat{u} exactly. In Sec.4 we formulate the main theorem for the case of uniform initial distribution concerning asymptotics of \hat{u} near the critical time T and the origin $x = 0$ and prove it. Here we present also the results of computations given directly according to the integral formula for \hat{u} that illustrate the fidelity of our asymptotic formulas. In Sec.5 we present certain results, both asymptotic and numeric, concerning a Gaussian initial distribution and show that for $0 \leq p < 1$ the behaviour of \hat{u} is close to the behaviour of the same function at $p = 0$, where the exact formula can be derived. In Sec.6 we discuss the question on the "observable" density and "induced" velocity. In Conclusion we sum up our results and discuss the question on a possibility to construct a PDE such that \hat{u} is its solution. We also argue possible applications of the results.

2. EXACT SOLUTION TO THE FOKKER-PLANCK EQUATION FOR SPECIAL INITIAL DATA

The Fokker-Planck equation associated to (1.8) for the probability density in position and velocity space $P = P(t, x, u)$ has the form

$$\frac{\partial P(t, x, u)}{\partial t} = \left[- \sum_{k=1}^n u_k \frac{\partial}{\partial x_k} + \beta \sum_{k=1}^n \left(u_k \frac{\partial}{\partial u_k} + 1 \right) + \frac{1}{2} \sigma^2 |u|^{2p} \frac{\partial^2}{\partial x_k^2} \right] P(t, x, u), \quad (2.1)$$

subject to the initial data

$$P(0, x, u) = P_0(x, u).$$

Let us set $\Omega_L := [-L, L]^n$, $L > 0$. Thus,

$$\hat{u}(t, x) = \lim_{L \rightarrow \infty} \frac{\int_{\Omega_L} u P(t, x, u) du}{\int_{\Omega_L} P(t, x, u) du}, \quad t \geq 0, x \in \Omega_L, \quad (2.2)$$

provided the limit exists.

If we choose

$$P_0(x, u) = \delta(u - u_0(x)) f(x) = \prod_{k=1}^n \delta(u_k - (u_0(x))_k) f(x), \quad (2.3)$$

with an arbitrary sufficiently regular $f(x)$, then

$$\hat{u}(0, x) = u_0(x).$$

The function $f(x)$ has the meaning of a probability density of the particle positions in the space at the initial moment of time.

Moreover, formula (2.2) can be investigated for functions $f(x)$ which are not necessarily probabilities densities.

Let us choose

$$u_0(x) = \alpha x, \quad \alpha < 0 \quad (2.4)$$

as initial data of the non-perturbed Burgers equation. One can see from (1.4), (1.5) that the gradient of the solution become unbounded as $t \rightarrow T$. Thus, we are interested in the behavior of $\hat{u}(t, x)$ comparing with the solution $u(t, x)$ to (1.1).

We apply formally the Fourier transform in the variable x to (2.1), (2.3) (2.4) to obtain for $\tilde{P} = \tilde{P}(t, \lambda, u)$

$$\frac{\partial \tilde{P}}{\partial t} = \beta \sum_{k=1}^n u_k \frac{\partial \tilde{P}}{\partial u_k} + \left(\beta - \frac{\sigma^2}{2} |u|^{2p} |\lambda|^2 - i(\lambda, u) \right) \tilde{P}, \quad (2.5)$$

$$\tilde{P}(0, \lambda, u) = \frac{1}{(|\alpha| \sqrt{2\pi})^n} e^{-i \frac{(\lambda, u)}{\alpha}} f\left(\frac{u}{\alpha}\right). \quad (2.6)$$

Equation (2.5) is of the first order, therefore the Cauchy problem (2.5), (2.6) for the function $\tilde{P}(t, \lambda, u)$ can easily be solved. Thus, for $\beta = 0$

$$\tilde{P}(t, \lambda, u) = \frac{f(u/\alpha)}{(|\alpha| \sqrt{2\pi})^n} e^{-\frac{\sigma^2}{2} |u|^{2p} |\lambda|^2 t - i(\lambda, u)(\frac{1}{\alpha} + t)}, \quad (2.7)$$

for $\beta > 0$

$$\tilde{P}(t, \lambda, u) = \frac{f(u e^{\beta t}/\alpha)}{(|\alpha| \sqrt{2\pi})^n} e^{\beta t - \frac{\sigma^2}{2p\beta} |u|^{2p} |\lambda|^2 (e^{\beta p t} - 1) - i(\lambda, u)((\frac{1}{\alpha} + \frac{1}{\beta}) e^{\beta t} - \frac{1}{\beta})}. \quad (2.8)$$

Further, the inverse Fourier transform gives for $\beta = 0$

$$P(t, x, u) = \frac{f(u/\alpha)}{(|\alpha|\sigma\sqrt{2\pi t})^n |u|^{pn}} e^{-\frac{|u(\frac{1}{\alpha}+t)-x|^2}{2\sigma^2 t |u|^{2p}}}, \quad t > 0, \quad (2.9)$$

and for $\beta > 0$

$$P(t, x, u) = \frac{f(u e^{\beta t}/\alpha) e^{-\beta t}}{(|\alpha|\sigma\sqrt{2\pi}(e^{2\beta p t} - 1)/\beta t)^n |u|^{pn}} e^{-\frac{2\beta p |u(e^{\beta t}(\frac{1}{\alpha} + \frac{1}{\beta}) - \frac{1}{\beta}) - x|^2}{2\sigma^2 |u|^{2p}(e^{2\beta p t} - 1)}}, \quad t > 0. \quad (2.10)$$

It is easy to see that the limit for $\beta \rightarrow 0$ of (2.10) gives (2.9).

Now we substitute $P(t, x, u)$ in (2.2) to get an integral representation of the quantity expectation $\hat{u}(t, x)$. It is important to note that for every even function $f(x)$ the upper integral in (2.2) vanishes at $x = 0$ or $t = T$, hence $\hat{u}(t, 0) = \hat{u}(T, 0) = 0$.

However, representation (2.2), (2.9) (or (2.10)) does not allow to see the behaviour of \hat{u} . Therefore we firstly consider the situations where the integrals in (2.2) can be explicitly computed (Sec.3) or analyze the asymptotics in certain important points (Sec.4).

3. CASE OF CONSTANT NOISE VARIANCE

Firstly we set $p = 0$. In this section we concentrate on a special choice of the initial probability density $f(x)$, allowing to obtain an explicit formula for $\hat{u}(t, x)$ if the initial data have the form (2.4).

Let us consider uniform initial distribution of particles as follow:

$$f(x) = \begin{cases} f_L = \frac{1}{(2L)^n}, & x \in \Omega_L; \\ 0, & \text{otherwise.} \end{cases} \quad (3.1)$$

Applying (2.2), (2.9), (2.10) we readily calculate

$$\hat{u}(t, x) = \begin{cases} \frac{\alpha x e^{-\beta t}}{1 + \frac{\alpha}{\beta}(1 - e^{-\beta t})}, & t < T; \\ 0, & t = T \end{cases}, \quad (3.2)$$

for $\beta > 0$ and

$$\hat{u}(t, x) = \begin{cases} \frac{\alpha x}{1 + \alpha t}, & t < T \\ 0, & t = T \end{cases}, \quad (3.3)$$

for $\beta = 0$. We also see that (3.3) results in the limit $\beta \rightarrow 0$ from (3.2).

It is enough to compare these expressions with (1.4), (1.5) to see that the white noise with a constant variance σ^2 does not influence the mean of the velocity. Thus, at any point (T, x) , $x \neq 0$, the function $\hat{u}(t, x)$ has a discontinuity (as for the case without noise).

We can also compute the variance

$$\hat{v}(t, x) := \lim_{L \rightarrow \infty} \frac{\int_{\Omega_L} (u - \hat{u})^2 P(t, x, u) du}{\int_{\Omega_L} P(t, x, u) du}, \quad t \geq 0, \quad x \in \mathbb{R}^n. \quad (3.4)$$

We have

$$\hat{v}(t, x) = \frac{\sigma t}{(t + \frac{1}{\alpha})^2}, \quad \beta = 0,$$

$$\hat{v}(t, x) = \frac{\sigma t}{\left(e^{\beta t} \left(\frac{1}{\alpha} + \frac{1}{\beta}\right) - \frac{1}{\beta}\right)}, \quad \beta > 0.$$

Thus, as $t \rightarrow T$, the denominator of $\hat{v}(t, x)$ becomes zero and the possible values of $U(t)$ at any x dissipate over the space.

A behavior of the mean of velocity which contrasts with the previous one can be obtained, e. g., for $f(x) = \left(\frac{k}{\sqrt{\pi}}\right)^n \exp(-k^2 x^2)$, $k > 0$. It is easy to compute that in this case for $\beta = 0$ one has

$$\hat{u}(t, x) = \frac{(1 + \alpha t) \alpha x}{\alpha^2 t^2 + 2(k^2 \sigma^2 + \alpha) t + 1}, \quad (3.5)$$

$$\hat{v}(t, x) = \frac{\sigma^2 \alpha^2 t}{\alpha^2 t^2 + 2(k^2 \sigma^2 + \alpha) t + 1}.$$

The denominator does not vanish for any fixed x, α, σ , so $\hat{u}(t, x) \rightarrow 0$, $\hat{v}(t, x) \rightarrow 0$, $t \rightarrow \infty$, and $\hat{u}(t, x)$ is continuous at any point (T, x) . The minimal value of $\text{div} \hat{u}(t, x)$ is attained at the time $t_* = \frac{1}{\alpha} \left(\sqrt{\frac{2}{-\alpha}} \sigma k - 1 \right) < T$. Moreover, $t_* > 0$ only if $\sigma k < \sqrt{\frac{2}{-\alpha}}$.

Computations made for some special classes of $f(x)$ allow to suggest that a similar behavior is provided by the function $\hat{u}(t, x)$ if $f(x) = f(|x|)$ and $\int_{\mathbb{R}^n} |x|^2 f(|x|) dx < \infty$. For example, for the class of $f(|x|) = \frac{\text{const}}{(1+k^2 x^2)^s}$, $n = 1$ we have the analytical result

$$\hat{u}(t, x) \sim \frac{\alpha^2}{(2(s-1)-1)k^2} x (1 + \alpha t), \quad \text{as } x \rightarrow 0, \quad s \in \mathbb{N}, s \geq 2.$$

A numerical study suggests that for $n = 1$ as $x \rightarrow 0$

$$\text{for } s > 1 \quad \hat{u}(t, x) \sim \text{const } x (1 + \alpha t),$$

$$\text{for } s \in \left[\frac{1}{2}, 1\right] \quad \hat{u}(t, x) \sim \text{const } x,$$

$$\text{for } s < \frac{1}{2} \quad \hat{u}(t, x) \sim \frac{\text{const}}{1 + \alpha t} x.$$

Here the constants do not depend on t .

4. ASYMPTOTIC BEHAVIOR FOR UNIFORM INITIAL DISTRIBUTION

We consider again $f(x)$ given in (3.1). If $p > 0$, formula (2.2) does not allow to compute $\hat{u}(t, x)$ explicitly. Thus, we need to extract from this formula an information that allows us to answer the main question of this article: does the stochastic perturbation suppress the singularity formation?

We will show that there exists a critical value of the parameter, $p = 1$, such that for $p \leq 1$ the mean of the velocity behaves very closely to the velocity in the case $p = 0$ (as in the non stochastically perturbed case for $t < T$). In contrast, for $p > 1$, $\hat{u}(t, x)$ vanishes as $t \rightarrow T$.

We deal with the case $\beta = 0$ to avoid cumbersome formulas, the results for $\beta > 0$ will be qualitatively the same, we present them in Remark 4.1.

Theorem 4.1. *The mean $\hat{u}(t, x)$ of the random variable $U(t)$, given the position $X(t)$, where $X(t), U(t)$ solve the SDE (1.8), provided $\hat{u}(0, x) = \alpha x$, $\alpha < 0$, is given by formula (2.2), where $P(t, x, u)$ is given by (2.9), (2.10).*

If $\beta = 0$ and initially the particles are distributed uniformly in the sense (3.1), the asymptotic behaviour of $\hat{u}(t, x)$ for $t \rightarrow T$ can be analyzed explicitly.

Namely, for any $p \in [0, 1)$ the mean $\hat{u}(t, x)$, being equal to zero at any point $x \in \mathbb{R}^n$, $t = T$, is discontinuous at every such point if $x \neq 0$. More precisely,

- *for $p = 0$ the mean $\hat{u}(t, x)$ coincides with the solution to the problem (1.1), (2.4) for $t < T$;*
- *for $p \in (0, 1)$ the asymptotics*

$$\hat{u}(t, x) = \frac{\alpha}{1 - \frac{t}{T}} x + o\left(\frac{1}{1 - \frac{t}{T}}\right), \quad t \rightarrow T, \quad x \in \mathbb{R}^n,$$

takes place.

- *For $p \geq 1$ at any $x \in \mathbb{R}^n$, $x \neq 0$, $|\hat{u}(t, x)| \rightarrow 0$ as $t \rightarrow T$, however, $\text{div}_x \hat{u}(t, x) \rightarrow \infty$, $x = 0$, $t \rightarrow T$. More precisely, for any $x \neq 0$,*

$$\hat{u}(t, x) = -C |x|^{\frac{2(1-p)}{p}} x \left(1 - \frac{t}{T}\right) + o\left(1 - \frac{t}{T}\right), \quad t \rightarrow T.$$

where C is the positive constant given in (4.4), and for any $t \in [0, T)$

$$\hat{u}(t, x) = \frac{\alpha}{1 - \frac{t}{T}} x + o(|x|), \quad x \rightarrow 0.$$

Proof. Let us set $\varepsilon := 1 - \frac{1}{T} t$, where the critical time T is introduced in (1.6) ($\varepsilon \in (0, 1]$). We will write below $t(\varepsilon) = (1 - \varepsilon)T$.

Firstly we compute asymptotics of (2.2) near the critical time $t = T$ ($\varepsilon = 0$).

Let us fix $x \neq 0$. We have the following expansion as $\varepsilon \rightarrow 0$, $p > 1$

$$\int_{\Omega_L} u P(t(\varepsilon), x, u) du = f_L \frac{\omega_n}{\alpha} \left(\sqrt{-\frac{1}{2\sigma^2\pi\alpha}} \right)^n \int_0^{|\alpha L|} e^{\frac{\alpha|x|^2}{2\sigma^2|u|^{2p}}} \frac{1}{|u|^{(n+2)p-n-1}} d|u| \, x\varepsilon + o(\varepsilon). \quad (4.1)$$

$$\int_{\Omega_L} P(t(\varepsilon), x, u) du = f_L \omega_n \left(\sqrt{-\frac{1}{2\sigma^2\pi\alpha}} \right)^n \int_0^{|\alpha L|} e^{\frac{\alpha|x|^2}{2\sigma^2|u|^{2p}}} \frac{1}{|u|^{pn-n+1}} d|u| + O(\varepsilon), \quad (4.2)$$

where ω_n is the area of surface of $(n - 1)$ - dimensional sphere, $n \geq 2$; $\omega_1 = 2$. Recall that for fixed x these integrals are functions of ε only.

Both integrals in (4.1) and (4.2) converge as $L \rightarrow \infty$ and can be expressed through Gamma-functions, so according to (2.2) we get

$$\hat{u}(t, x) = \frac{\frac{1}{\alpha} \left(\sqrt{-\frac{1}{2\sigma^2\pi\alpha}} \right)^n \int_0^\infty e^{\frac{\alpha|x|^2}{2\sigma^2|u|^{2p}}} \frac{1}{|u|^{(n+2)p-n-1}} d|u| \, x\varepsilon + o(\varepsilon)}{\left(\sqrt{-\frac{1}{2\sigma^2\pi\alpha}} \right)^n \int_0^\infty e^{\frac{\alpha|x|^2}{2\sigma^2|u|^{2p}}} \frac{1}{|u|^{pn-n+1}} d|u| + O(\varepsilon)} =$$

$$\begin{aligned}
& \frac{\varepsilon x |x|^{(n+2)\frac{1-p}{p}} \frac{1}{\alpha} \left(\sqrt{-\frac{1}{2\sigma^2\pi\alpha}} \right)^n \left(\sqrt{-\frac{\alpha}{2\sigma}} \right)^{(n+2)\frac{1-p}{2p}} \Gamma\left((n+2)(p-1)/2p\right) + o(\varepsilon)}{|x|^{n(1-p)/p} \left(\sqrt{-\frac{1}{2\sigma^2\pi\alpha}} \right)^n \left(\sqrt{-\frac{\alpha}{2\sigma^2}} \right)^{n\frac{1-p}{2p}} \Gamma\left(n(p-1)/2p\right) + O(\varepsilon)} = \\
& = -C \varepsilon x |x|^{\frac{2(1-p)}{p}} + o(\varepsilon), \quad \varepsilon \rightarrow 0,
\end{aligned} \tag{4.3}$$

with the positive constant

$$C = -\frac{1}{\alpha} \left(\sqrt{-\frac{\alpha}{2\sigma^2}} \right)^{\frac{1-p}{2p}} \frac{\Gamma\left((n+2)(p-1)/2p\right)}{\Gamma\left(n(p-1)/2p\right)}. \tag{4.4}$$

If $p \leq 1$, the integrals in (4.1) and (4.2) diverge as $L \rightarrow \infty$. Now we take into account that

$$\begin{aligned}
\int_{\Omega_L} P(t(\varepsilon), x, u) du &= f_L \int_0^{|\alpha L|} \sum_{k=0}^{\infty} F_k(x, |u|) \varepsilon^k d|u|, \\
F_k(x, |u|) &= e^{\frac{\alpha|x|^2}{2\sigma^2|u|^{2p}}} \sum_{s=0}^k \frac{c_{ks} |x|^{2s}}{|u|^{sp+np-n+1}}, \\
\int_{\Omega_L} u P(t(\varepsilon), x, u) du &= \varepsilon x \int_0^{|\alpha L|} \sum_{k=0}^{\infty} G_k(x, |u|) \varepsilon^k d|u|, \\
G_k(x, |u|) &= e^{-\frac{\alpha|x|^2}{2\sigma^2|u|^{2p}}} \sum_{s=0}^k \frac{b_{ks} |x|^{2s}}{|u|^{(s+1)p+np-n-1}},
\end{aligned}$$

with some constant b_{ks} and c_{ks} . Further,

$$\begin{aligned}
\hat{u}(t(\varepsilon), x) &= \varepsilon x \lim_{L \rightarrow \infty} \frac{\int_0^{|\alpha L|} \sum_{k=0}^{\infty} G_k(x, \xi) \varepsilon^k d\xi}{\int_0^{|\alpha L|} \sum_{k=0}^{\infty} F_k(x, \xi) \varepsilon^k d\xi} = \\
&= x \varepsilon \lim_{L \rightarrow \infty} \frac{\frac{\omega_n}{\alpha} \left(\sqrt{-\frac{1}{2\sigma^2\pi\alpha}} \right)^n \int_0^{|\alpha L|} e^{\frac{\alpha|x|^2}{2\sigma^2\xi^{2p}}} \frac{1}{\xi^{(n+2)p-n-1}} d\xi (1 + O(\varepsilon))}{\omega_n \left(\sqrt{-\frac{1}{2\sigma^2\pi\alpha}} \right)^n \int_0^{|\alpha L|} e^{\frac{\alpha|x|^2}{2\sigma^2\xi^{2p}}} \frac{1}{\xi^{np-n+1}} d\xi (1 + O(\varepsilon))} = \\
&= \begin{cases} \frac{x\varepsilon}{\alpha} + o(\varepsilon), & p = 1, \\ -\infty \operatorname{sgn} x, & p < 1. \end{cases},
\end{aligned} \tag{4.5}$$

where we use de L'Hôpital's rule to compute the ratio of the divergent integrals in (4.5).

Thus, for $p = 1$ we get (4.3) again.

For $p < 1$ we are going to obtain a more specified result. Recall that

$$\begin{aligned}
& P(t(\varepsilon), x, u) = \\
& = f\left(\frac{u}{\alpha}\right) \left(\frac{1}{\left(\frac{2\pi\sigma^2(\varepsilon-1)}{\alpha}\right)^{n/2}} \frac{\exp\left(-\frac{|u|^{2-2p}\varepsilon^2}{2\alpha\sigma^2(\varepsilon-1)}\right)}{|u|^{np}} \right) \left(\exp\left(-\frac{\alpha(|x|^2 - 2(x, u)\varepsilon/\alpha)}{2\sigma^2(\varepsilon-1)|u|^{2p}}\right) \right).
\end{aligned} \tag{4.6}$$

For the third factor as $\varepsilon \rightarrow 0$ we have:

$$e^{\frac{2\alpha|x|^2}{\sigma^2|u|^{2p}}} + e^{\frac{2\alpha|x|^2}{\sigma^2|u|^{2p}}} \left(-\frac{(x, u)}{\sigma^2|u|^{2p}} + \frac{\alpha|x|^2}{2\sigma^2|u|^{2p}} \right) \varepsilon + O(\varepsilon^2).$$

The second factor in (4.6) secures the convergence of integrals. Calculations of both integrals in (2.2) with the use of the Maple environment allow to obtain explicit formulas for rational p , the result being expressed through special functions (Bessel, Gamma and hypergeometric functions). We do not quote here this formula as it is very cumbersome. The simplest result is for $p = \frac{1}{2}$:

$$\lim_{L \rightarrow \infty} \int_{\Omega_L} P(x, u, t) du = 2 f_L K \left(\frac{n}{2}, \frac{\varepsilon|x|}{\sigma^2 \sqrt{1-\varepsilon}} \right) \left(\frac{|x|}{2\sigma^2 \pi \varepsilon \sqrt{1-\varepsilon}} \right)^{\frac{n}{2}} (1 + O(\varepsilon))$$

$$\begin{aligned} & \lim_{L \rightarrow \infty} \int_{\Omega_L} u P(x, u, t) du = \\ & = f_L \frac{2\alpha\sqrt{1-\varepsilon}x}{n\sigma^2} K \left(\frac{n}{2} + 1, \frac{\varepsilon|x|}{\sigma^2 \sqrt{1-\varepsilon}} \right) \left(\frac{|x|}{2\sigma^2 \pi \varepsilon \sqrt{1-\varepsilon}} \right)^{\frac{n}{2}} (1 + O(\varepsilon)), \end{aligned}$$

where K is the modified Bessel function of the second kind [13]. Thus,

$$\hat{u}(t, x) = \frac{2\alpha\sqrt{1-\varepsilon}x}{n\sigma^2} \frac{K \left(\frac{n}{2} + 1, \frac{\varepsilon|x|}{\sigma^2 \sqrt{1-\varepsilon}} \right)}{K \left(\frac{n}{2}, \frac{\varepsilon|x|}{\sigma^2 \sqrt{1-\varepsilon}} \right)} (1 + O(\varepsilon)).$$

Calculations using asymptotic expansion of the K functions for $\varepsilon \rightarrow 0$ show that

$$\hat{u}(t, x) = \frac{\alpha}{\varepsilon} x + O(1), \quad \varepsilon \rightarrow 0.$$

Analogously result we get in the case of any rational $p < 1$. Namely, if we assume $p = \frac{m_1}{m_2}$, $m_1, m_2 \in \mathbb{N}$, we obtain

$$\hat{u}(t, x) = \frac{\alpha}{\varepsilon} x + o\left(\frac{1}{\varepsilon}\right), \quad \varepsilon \rightarrow 0. \quad (4.7)$$

In particular, for $p = \frac{1}{m}$, $m \in \mathbb{N}$

$$\hat{u}(t, x) = \frac{\alpha}{\varepsilon} x + O\left(\varepsilon^{-\frac{2m-3}{2m-1}}\right), \quad \varepsilon \rightarrow 0.$$

Then, we can consider p as a parameter and notice that at any fixed $x \neq 0$, $\varepsilon \in (0, 1]$ the functions $P(t(\varepsilon), x, u; p)$ and $uP(t(\varepsilon), x, u; p)$ are continuous on the set $u \in \Omega_L$, $p \in [0, 1]$. Moreover, $\frac{1}{f_L} \lim_{L \rightarrow \infty} \int_{\Omega_L} P(t(\varepsilon), x, u; p) du$ and $\frac{1}{f_L} \lim_{L \rightarrow \infty} \int_{\Omega_L} u P(t(\varepsilon), x, u; p) du$ are uniformly bounded for $p \in [0, 1 - \delta]$, δ is a positive arbitrary small constant (see (4.6)). Thus, by a standard reasoning we conclude that the ratio $\hat{u}(t, x)$ is continuous in p , $p \in [0, 1 - \delta]$. Since there exist a rational sequence convergent to every real p , the continuity implies that (4.7) holds for all $p \in [0, 1 - \delta]$. Over the arbitrariness of δ we can conclude that (4.7) is true for all $p \in [0, 1]$.

Now we fix $\varepsilon > 0$ and expand the integrals in (2.2) near $x = 0$. We have for $\beta = 0$, any $k = 1, \dots, n$, using (2.9):

$$\int_{\Omega_L} u_k P(t(\varepsilon), x, u) du =$$

$$\begin{aligned}
&= \frac{f_L \varepsilon x_k}{\sigma^2(\varepsilon - 1) (2\pi\sigma^2(\varepsilon - 1)\alpha)^{n/2}} \omega_n \times \\
&\times \int_0^{|\alpha L|} \exp\left(-\frac{|u|^{2-2p} \varepsilon^2}{2\alpha\sigma^2(\varepsilon - 1)}\right) |u|^{n(1-p)-1-2p} u_k^2 d|u| + o(|x|^2) = \\
&= c_k(\varepsilon) x_k,
\end{aligned}$$

for some functions $c_k(\varepsilon)$. Moreover,

$$\begin{aligned}
&\int_{\Omega_L} P(t(\varepsilon), x, u) du = \\
&\frac{f_L \omega_n}{(2\pi\sigma^2(\varepsilon - 1)\alpha)^{n/2}} \int_0^{|\alpha L|} \exp\left(-\frac{|u|^{2-2p} \varepsilon^2}{2\alpha\sigma^2(\varepsilon - 1)}\right) |u|^{n(1-p)-1} d|u| + o(|x|) = \\
&= \frac{f_L}{2|p-1|} (\varepsilon^2 \pi)^{-\frac{n}{2}} \Gamma\left(\frac{n}{2}\right) + o(|x|). \tag{4.8}
\end{aligned}$$

All integrals converge for $p > 0$, $p \neq 1$ as $L \rightarrow \infty$. Further,

$$A(\varepsilon) := \sum_{j=k}^n c_k(\varepsilon) = \frac{\alpha}{\varepsilon|p-1|} (\varepsilon^2 \pi)^{-\frac{n}{2}} \Gamma\left(1 + \frac{n}{2}\right),$$

due to the equality of space directions $c_k(\varepsilon) = \frac{1}{n} A(\varepsilon)$, therefore

$$\hat{u}(t, x) = \frac{\alpha}{\varepsilon} x + o(|x|), \quad |x| \rightarrow 0. \tag{4.9}$$

In the limit case $p = 1$, where both integrals in the ratio (2.2) diverge, we can apply de L'Hôpital's rule as above (in (4.9) to obtain (4.6)).

Thus, the theorem is proved. \square

Remark 4.1. For $\beta > 0$, $p \geq 1$ we get the expansion (4.3) as $\varepsilon \rightarrow 0$ again, but in this case

$$C = -\frac{1}{\beta} \ln \frac{\alpha + \beta}{\alpha} \left(2 \left(\left(\frac{\alpha}{\beta + \alpha} \right)^{2p} - 1 \right) \frac{\sigma}{2p\beta} \right)^{\frac{p-1}{p}} \frac{\Gamma((n+2)(p-1)/2p)}{\Gamma(n(p-1)/2p)},$$

for $p \in [0, 1)$ we get qualitatively the same result as in (4.7).

Analogously we have for any $p > 0$, $\varepsilon \in (0, 1]$

$$\hat{u}(t, x) = \frac{\beta}{\left(\frac{\beta}{\alpha} + 1\right)^\varepsilon - 1} x + o(|x|), \quad |x| \rightarrow 0.$$

Remark 4.2. Let us compute the conditional variance $\hat{v}(t, x)$. For $p > 1 + \frac{4}{n}$ we can expand the probability density $P(t, u, x)$ in ε near $\varepsilon = 0$ and obtain that for any $x \neq 0$

$$\hat{v}(t(\varepsilon), x) = F(n, p, \alpha, \sigma) |x|^{\frac{2}{p}} \left(1 - \frac{n(p-1) - (p+2)}{2p} \varepsilon \right) + o(\varepsilon),$$

with

$$F(n, p, \alpha, \sigma) = \frac{\Gamma\left(\frac{n(p-1)-2}{2p}\right)}{\Gamma\left(\frac{n(p-1)}{2p}\right)} \left(-\frac{\alpha}{4\sigma^2}\right)^{\frac{1}{p}}.$$

Thus, at any fixed $x \neq 0$ the variance tends to some finite value as $\varepsilon \rightarrow 0$.

Further, for any fixed $\varepsilon > 0$ we get the following asymptotic expansion near $x = 0$:

$$\hat{v}(t(\varepsilon), x) = \frac{\Gamma\left(\frac{n(p-1)-2}{2p}\right)}{\Gamma\left(\frac{1}{2}n\right)} \left(-\frac{\varepsilon^2}{4\alpha\sigma^2(1-\varepsilon)}\right)^{\frac{1}{p-1}} + o(|x|),$$

therefore the variance tends to zero as $x \rightarrow 0$ and $\varepsilon \rightarrow 0$.

We cannot write an explicit formula for the variance for all $p \leq 1 + \frac{4}{n}$. However, for $p \leq 1$ in the expression for the second moment both integrals in the ratio diverge as $L \rightarrow \infty$, and we can use de L'Hôpital's rule once more to show that the variance tends to infinity as $\varepsilon \rightarrow 0$. In particular, for $p = \frac{1}{m}$, $m \in \mathbb{N}$,

$$\hat{v}(t, x) = O\left(\varepsilon^{-\frac{4m}{2m-1}}\right), \quad \varepsilon \rightarrow 0.$$

5. ASYMPTOTIC BEHAVIOR FOR A GAUSSIAN INITIAL DISTRIBUTION

Let us set $f = \left(\frac{k}{\sqrt{\pi}}\right)^n \exp(-k^2 x^2)$, $k > 0$, and study the asymptotics of (2.2), (2.9) near $x = 0$. We consider only the case $\beta = 0$ and $n = 1$. As we have seen in Sec.3, this type of initial density distribution eliminates the unbounded gradient growth as $\varepsilon \rightarrow 0$ in the case of a constant σ .

To get a qualitative result we firstly find an asymptotic expansion near $x = 0$ for both integrals in (2.2), and for their ratio in ε at the point $\varepsilon = 0$ ($t = T$). The computations made in the environment Maple show that for $p < 1$

$$\hat{u} = -C\varepsilon x + o(\varepsilon)C_1(x) + o(|x|)C_2(\varepsilon),$$

and for $p \geq 1$

$$\hat{u} = -\frac{\alpha}{\varepsilon}x + Cx + o(\varepsilon)C_1(x) + o(|x|)C_2(\varepsilon),$$

with a positive constant C depending only on α, σ, k and bounded functions C_1 and C_2 .

Numerical computations show that if $p < 1$ the function $\hat{u}(t, x)$ is very close to the respective function for $p = 0$. In particular, the singularity at the origin is also eliminated. If $p \geq 1$, the picture is similar to the case of uniform initial distribution $f(x)$.

6. "OBSERVABLE" DENSITY AND "INDUCED" VELOCITY

Introduce the function

$$\hat{\rho}(t, x) = \lim_{L \rightarrow \infty} \int_{\Omega_L} P(t, x, u) du.$$

If $\sigma = 0$, in (2.1), then $\hat{\rho}(t, x)$ solves the continuity equation

$$\frac{\partial \hat{\rho}}{\partial t} + \operatorname{div}(u \hat{\rho}) = 0, \quad (6.1)$$

and therefore corresponds to the distribution of density for the particles provided initial data $\hat{\rho}(0, x) = f(x) > 0$ are given. Here we do not require that $f(x)$ is a

probability density. Thus, $\hat{\rho}(t, x)$ coincides with the real density of particles that we denote by $\rho(t, x)$.

For $\sigma \neq 0$ the function $\hat{\rho}(t, x)$ is not a solution to (6.1). Nevertheless an observer who does not know the random distribution of the particles positions considers $\hat{\rho}(t, x)$ as the density for the particles. We will call $\hat{\rho}(t, x)$ in this case an observable density. It differs from $\rho(t, x)$. For example, it is easy to compute that for $f(x) = C$ and $\beta = 0$ from the system (1.1), (6.1) one gets $\rho(t, x) = \frac{C}{(1+\alpha t)^n} = C \varepsilon^{-n}$. The observable density $\hat{\rho}(t, x)$, as follows from (4.8), behaves in a quite different way. The only common feature is the asymptotics $O(\varepsilon^{-n})$ as $|x| \rightarrow 0$. Nevertheless, given a density ρ , one can find from (6.1) the velocity $u(t, x)$. Let us consider the case $n = 1$. If we assume

$$\lim_{|x| \rightarrow \infty} \rho(t, x)u(t, x) = 0, \quad t \in \mathbb{R}_+$$

(this implies the momentum conservation), we get

$$u(t, x) = -\frac{\int_{-\infty}^x \rho'_t(t, x) dx}{\rho(t, x)}.$$

Remark that $\hat{\rho}(t, x)$ is positive for $f(x) > 0$. Let us introduce the vector-function $v(t, x)$ according to the formula

$$v(t, x) = -\frac{\int_{-\infty}^x \hat{\rho}'_t(t, x) dx}{\hat{\rho}(t, x)}. \quad (6.2)$$

It is natural to call $v(t, x)$ the "induced" velocity. As follows from (6.2), (2.1), we have

$$v(t, x) = \hat{u}(t, x) + v_1(t, x), \quad (6.3)$$

where

$$v_1(t, x) := \frac{\sigma^2}{2} \frac{\int_{\mathbb{R}} |u|^{2p} P'_x(t, x, u) du}{\int_{\mathbb{R}} P(t, x, u) du}.$$

One can see that if $P(t, x, u)$ is given by (2.9) in the case $f(x) = \text{const}$ the second term v_1 in (6.3) vanishes at $x = 0$. Moreover, computations show that as $|x| \rightarrow 0$

$$v_1(t, x) = C(t) \varepsilon x |x|^2 + o(|x|^3),$$

with a function $C(t)$ which stays bounded as $\varepsilon \rightarrow 0$. Thus, in the vicinity of the point of the singularity formation the induced velocity is close to \hat{u} .

7. CONCLUSION

We have shown that in the model (1.8), generally speaking, the expectation of the velocity given position differs drastically from the velocity in the Burgers equation in the deterministic coordinate space. Under the observation in the random position of particles the blow up phenomena can be lost. In this context the uniform initial distribution of the particles position seems most interesting. Here the threshold effect arises. Namely, if the exponent p is less or equal then 1, then the expectation \hat{u} follows the real behavior of the velocity rather well. However, if $p \geq 1$, the function \hat{u} tends to zero whereas the real velocity tends to infinity at any point outside the origin $x = 0$ as the time tends to the critical value.

The question on a PDE obeyed by the function \hat{u} arises naturally, due to existing formalisms to represent solutions of PDE as the expected value of functionals of stochastic processes (see e.g. [14],[15], [16] and references therein). In [17] one can find a recent result in the field close to our study, the stochastic formulation of the viscous Burgers equation. Namely, it was shown that if the pair (u, X) (where X is the flow map such that $u \circ X$ is constant in time) solve the stochastic system

$$dX = u dt + \sqrt{2\nu} dW, \quad u = \mathbf{E}[u_0 \circ X^{-1}], \quad X(x_0, 0) = x_0, \quad \nu = \text{const} > 0,$$

then under certain additional conditions u satisfies the viscous Burgers equation

$$u_t + (u, \nabla) u = \nu \Delta u. \quad (7.1)$$

The function \hat{u} also solve (7.1) for $p = 0$, however only for the special initial data and uniform initial distribution of particles. For example, one can check that \hat{u} defined as (2.2), with $P(t, x, u)$ governed by (2.1) does not satisfies the viscous Burgers equation even for $p = 0$ and linear initial data for the Gaussian initial distribution (see (3.5)). Nevertheless, the comparison of \hat{u} and the solution to the Burgers equation with a specific viscosity term is a very interesting open question.

At last we would like to discuss possible applications of the results of this paper. First of all the the stochastic systems like (1.8) with velocity field of the fluid, with possibly random component with prescribed statistics can be applied to model turbulent or other disordered fluctuations. For example, the problem can be in describing some desired statistics of the trajectory $X(t)$ of a tracer particle released initially from some point x_0 and subsequently transported jointly by the flow $u(x, t)$ and molecular diffusivity [18].

Other applications can be found in the theoretical financial mathematics to model the earnings yield of a risky asset $(X(t))$ with the trend and volatility depending on a certain macroeconomic factor $U(t)$ (e.g. the spot interest rate) in the spirit of [19]. Let us notice in this context that for $p = 0$ in several cases it is possible to solve the Fokker-Planck equation for more general class of stochastic DE, describing the economic quantities in a more agreed way, namely

$$dX_k(t) = (a_k + A_{ki}U_i(t)) dt + \sum_{i=1}^{m+n} \sigma_{ki} d(W_i)_t, \quad k = 1, \dots, n;$$

$$dU_i(t) = (b_i + B_{ik}U_k(t)) dt + \sum_{k=1}^{m+n} \lambda_{ik} d(W_k)_t, \quad i = 1, \dots, m;$$

$$X_k(0) = x_k, \quad U_i(0) = u_i, \quad t \geq 0,$$

where $W(t)$ is a \mathbb{R}^{m+n} valued standard Brownian motion process, the market parameters $a_k, A_{ik}, \sigma_{ik}, b_k, B_{ik}, \lambda_{ik}$ are constant matrices of appropriate dimensions.

ACKNOWLEDGMENTS

Stimulating discussion with Y.I.Belopolskaya and V.M.Shelkovich are gratefully acknowledged. We thank M.Freidlin and L.Ryzhik for attracting attention to the paper [17].

REFERENCES

- [1] E.Hopf, The partial differential equation $u_t + uu_x = \mu u_{xx}$, *Comm.Pure Appl.Math* **3** (1950) 201-230.
- [2] S.Albeverio, A.Klar, Long time behavior of nonlinear stochastic oscillators: The one- dimensional Hamiltonian case. *J. Math. Phys.* **35** (8) (1994) 4005-4027.
- [3] S.Albeverio, A. Klar, Longtime behaviour of stochastic Hamiltonian systems: the multidimensional case. *Potential Anal.* **12** (3) (2000) 281-297.
- [4] S.Albeverio, V.N. Kolokoltsov, The rate of escape for some Gaussian processes and the scattering theory for their small perturbations. *Stochastic Processes and their Applications* **67** (2) (1997) 139-159.
- [5] S. Albeverio, A. Hilbert, E. Zehnder, Hamiltonian systems with a stochastic force: nonlinear versus linear and a Girsanov formula. *Stochastics and Stochastics Reports* **39** (1992), 159-188.
- [6] H. Risken, *The Fokker-Planck Equation Methods of Solution and Applications*, Second Edition, (Springer-Verlag, 1989).
- [7] W.A. Woyczyński, *Burgers-KPZ Turbulence*, LNM 1700, Springer, 1998.
- [8] J. Bec, K.Khanin, Burgers turbulence, *Physics Reports* **447** (1) (2007), 1 - 66 .
- [9] J.P.Bouchaud, M.Mezard, Velocity fluctuations in forced Burgers turbulence, *Phys.Rev. E* **54** (1996) 5116
- [10] V.Gurarie, Burgers equations revisited, *arXiv:nlin/0307033v1 [nlin.CD]* (2003).
- [11] B.Øksendal, *Stochastic differential equations: an introduction with applications*, 5th ed.(Springer-Verlag Heidelberg New York, 1998).
- [12] A.J.Chorin, O.H.Hald, *Stochastic tools in mathematics and science* (New York: Springer, 2006).
- [13] I.S. Gradshteyn, I.M. Ryzhik, *Table of integrals, series, and products*, 6th ed. (San Diego, CA: Academic Press, 2000).
- [14] M.Freidlin, *Functional integration and partial differential equations*, Annals of Mathematics Studies, No.109 (Princeton, New Jersey: Princeton University Press, 1985).
- [15] M.Freidlin, *Markov processes and differential equations: asymptotic problems*, Lectures in Mathematics, (ETH Zu"rich. Basel: Birkha"user, 1996).
- [16] Ya.I.Belopolskaya; Yu.L.Daletskij, *Stochastic equations and differential geometry* (Kluwer Academic Publishers, 1990).
- [17] P.Constantin, G.Iyer, A stochastic Lagrangian representation of the three-dimensional incompressible Navier-Stokes equations. *Commun. Pure Appl. Math.* **61** (3) 330-345 (2008).
- [18] A.J. Majda, P.R. Kramer, Simplified models for turbulent diffusion: Theory, numerical modelling, and physical phenomena *Physics Reports* **314** 237-574 (1999).
- [19] T.R.Bielecki, S.R.Pliska, Risk-sensitive dynamic asset management. *Appl. Math. Optimization* **39** (3) 337-360 (1999).

(¹) UNIVERSITÄT BONN, INSTITUT FÜR ANGEWANDTE MATHEMATIK, ABTEILUNG FÜR STOCHASTIK, WEGELERSTRASSE 6, D-53115, BONN; IZKS, BONN; BiBoS, BIELEFELD-BONN

(²) MATHEMATICS AND MECHANICS FACULTY, MOSCOW STATE UNIVERSITY, MOSCOW 119992, RUSSIA

E-mail address, ¹: albeverio@uni-bonn.de

E-mail address, ²: rozanova@mech.math.msu.su